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Methods for estimating Weibull parameters for brittle materials

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Abstract A Monte Carlo simulation is used to obtain the statistical properties of the Weibull parameters estimated by the linear regression, weighted linear regression, maximum likelihood and moments methods, respectively. Results reveal that the estimated Weibull modulus is always biased, which has a much lower accuracy than the scale parameter. The mean square error is adopted as a criterion for the comparison of the estimation methods. It is shown that both the probability estimators and the weight factors have great effects on the estimation precision of the Weibull modulus. The weighted linear regression with a weight factor of W_i=3.3P_i – 27.5[1–(1–P_i)$^{0.025}$] and a probability estimator of P_i=(i–0.3)/(n+0.4) gives the most accurate estimation for sample sizes of 9–52. The maximum likelihood method recommended for any sample size by previous authors, comes first only for sample sizes larger than or equal to 53; furthermore, it is less conservative than the regression methods, and hence results in a lower safety in reliability predictions.

Introduction

Weibull statistics has been commonly used to characterize the statistical variation in the fracture strength of brittle materials such as ceramics, glasses and solid catalysts [1–5]. It is based on a ''weakest link theory'', which means that the most serious flaw in the material will control the strength, like a chain breaking if the weakest link fails. The most serious flaw is not necessarily the largest one because its severity also relies on its location and orientation. In other words, the flaw subjected to the highest stress intensity factor will be strength controlling.

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Using Weibull's two-parameter distribution, the probability of failure P at or below a stress σ is represented by [1, 6, 7]

$$P = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \quad (1)$$

where m and σ_0 are the Weibull modulus and the scale parameter, respectively. The Weibull modulus m, also called the shape parameter, represents the scatter in the fracture strength. A higher m leads to a steeper distribution function and thus a lower dispersion of the fracture strength. The scale parameter σ_0 corresponding to the fracture stress with a failure probability of 63.2% is closely related to the mean strength $\bar{\sigma}$ of the distribution [1, 7].

$$\bar{\sigma} = \sigma_0 \Gamma\left(1 + \frac{1}{m}\right) \quad (2)$$

where Γ is the gamma function. For the Weibull modulus of 5–20, a typical range for technical ceramics [7], $\Gamma(1+1/m)$ takes values between 0.9 and 1.

There are several methods available in the literature [7–18], for the determination of the Weibull distribution parameters from a set of experimentally measured fracture stresses. However, the values of the estimated Weibull parameters can vary according to the method employed. The interest of choosing an appropriate method for an accurate estimation of the Weibull parameters, therefore, arises. Bergman [8] and Sullivan and Lauzon [9] investigated the relative merits of four commonly-used probability estimators in the linear regression method by using computer-generated data and actual experimental results, respectively. Khalili and Kromp [7] and Trustrum and Jayatilaka [10] compared the linear regression, the maximum likelihood and the moments methods based on a Monte Carlo simulation. Bergman [11], Faucher and Tyson [12] and Langlois [13] reported that in the linear regression, the use of the weight factors could improve the estimation quality of the Weibull parameters.

In spite of several contributions on the subject, it seems that a common mistake has been made as to the selection of the criterion for the comparison of the estimation methods. In this paper, the common methods for estimating Weibull parameters will first be outlined, and then the effectiveness of these methods is compared based on a minimum mean-square-error criterion which has never been used in previous studies. A Monte Carlo simulation is applied to obtain the statistical properties of the estimated Weibull parameters. It is found that the obtained results are very different from those reported by previous authors.

Estimation of the Weibull parameters

Linear regression (LR) method

The most widely used method may be the linear regression analysis due to its simplicity. The measured fracture stresses are ranked in ascending order and then a probability of failure P_i is assigned to each stress σ_i. Since the true value of P_i is unknown, a prescribed estimator has to be used. The following four expressions are often applied to define the probability estimator [8, 9, 14–18].

$$P_i = \frac{i - 0.5}{n} \tag{3a}$$

$$P_i = \frac{i}{n + 1} \tag{3b}$$

$$P_i = \frac{i - 0.3}{n + 0.4} \tag{3c}$$

$$P_i = \frac{i - 3/8}{n + 1/4} \tag{3d}$$

where P_i is the probability of failure for the ith ranked stress datum, and n is the sample size.

By taking the logarithm twice, Eq. (1) can be rewritten in a linear form.

$$\ln\ln\left(\frac{1}{1 - P}\right) = m\ln\sigma - m\ln\sigma_0 \tag{4}$$

The Weibull modulus can thus be obtained directly from the slope term in Eq. (4) and the scale parameter can be deduced from the intercept term.

Weighted linear regression (WLR) method

As proposed by Bergman [11], assuming the same weight for each datum point of Eq. (4) is erroneous, and an appropriate weight function needs to be applied to improve the conventional LR.

Equation (4) can be reduced to

$$Y = a + bX \tag{5}$$

where $Y = \ln\ln 1/(1 - P), X = \ln\sigma, \ a = -m\ln\sigma_0$, and b=m. The WLR is based on a hypothesis that a straight line fitting must minimize the weighted sum of squares of deviations of the data Y_i from the fitting function $Y(X_i)$, i.e. the equation

$$k^2 = \sum W_i(Y_i - a - bX_i)^2 \tag{6}$$

takes the minimum value. By putting $\partial k^2/\partial a$=$\partial k^2/\partial b$=0, it results in

$$m = b = \frac{\sum W_i \sum X_i Y_i W_i - \sum Y_i W_i \sum X_i W_i}{\sum W_i \sum X_i^2 W_i - \left(\sum X_i W_i\right)^2} \tag{7}$$

$$-m\ln\sigma_0 = a = \frac{\sum Y_i W_i - m\sum X_i W_i}{\sum W_i} \tag{8}$$

where W_i is the weight factor for the ith datum point. Two Weibull parameters can be calculated from Eqs. (7) and (8). It is clear that the conventional LR is a special case of the WLR at W_i=1.

Bergman [11] derived a weight factor based on the theory of error propagation.

$$W_i = [(1 - P_i)\ln(1 - P_i)]^2 \tag{9a}$$

Faucher and Tyson [12] considered this question further and proposed another weight factor which can be approximated by

$$W_i = 3.3P_i - 27.5\left[1 - (1 - P_i)^{0.025}\right] \tag{9b}$$

Similarly to the conventional LR, the probability of failure for each strength datum ranked in ascending order is also approximated by Eqs. (3a–d).

Maximum likelihood (ML) method

The method of maximum likelihood is another often used procedure for estimating Weibull parameters. In this procedure, the Weibull parameters, m and σ_0, are sought, which results in a Weibull function describing most likely the experimental data. A likelihood function L is defined as:

$$L = \prod_{i=1}^{n} f(\sigma_i) \tag{10}$$

where

$$f(\sigma_i) = dP_i/d\sigma_i = \frac{m}{\sigma_0}\left(\frac{\sigma_i}{\sigma_0}\right)^{m-1} \exp\left[-\left(\frac{\sigma_i}{\sigma_0}\right)^{m}\right] \tag{11}$$

Equation (10) should be maximized, which requires $\partial \ln L/\partial m = \partial \ln L/\partial \sigma_0 = 0$. The derivation is done on the logarithm of L, since taking the derivation of a sum is easier than that of a product. The detailed calculation can be found in many papers [19–21]. The result is:

$$n/m - n\sum_{i=1}^{n}\left(\sigma_i^m \ln \sigma_i\right)/\sum_{i=1}^{n}\sigma_i^m + \sum_{i=1}^{n}\ln \sigma_i = 0 \tag{12}$$

$$\sigma_0 = \left(\sum_{i=1}^{n}\sigma_i^m/n\right)^{1/m} \tag{13}$$

Although Eq. (12) is non-linear, it has a unique positive solution for m [21], and may be solved by standard iterative procedures, e.g. the Newton–Rhapson method. Once m is determined, the scale parameter σ_0 can be estimated using Eq. (13).

Moments method

The first and the second moments of the Weibull distribution result in the mean value $\bar{\sigma}$ given by Eq. (2) and the variance S^2 of the distribution, respectively.

$$S^2 = \sigma_0^2\left[\Gamma\left(1+\frac{2}{m}\right) - \Gamma^2\left(1+\frac{1}{m}\right)\right] \tag{14}$$

The standard deviation S is the square root of the variance. By dividing S by $\bar{\sigma}$, one can get the coefficient of variation CV of the Weibull distribution, which is a function of m only

$$CV = \frac{S}{\bar{\sigma}} = \left[\Gamma\left(1+\frac{2}{m}\right) - \Gamma^2\left(1+\frac{1}{m}\right)\right]^{1/2} \Big/ \Gamma\left(1+\frac{1}{m}\right) \tag{15}$$

In the method of moments, it is assumed that the sample moments equal those of the distribution. Setting the mean and standard deviation of the experimentally measured data in Eq. (15), the Weibull modulus can therefore be determined using the Newton–Rhapson method. And the scale parameter may be calculated from the transformation form of Eq. (2) as follows:

$$\sigma_0 = \bar{\sigma} \Big/ \Gamma\left(1+\frac{1}{m}\right) \tag{16}$$

Monte Carlo simulation

Equation (1) can be rewritten as

$$\sigma = \sigma_0\left[\ln\left(\frac{1}{1-P}\right)\right]^{1/m} \tag{17}$$

If we consider a large ''specimen'' population with prescribed m and σ_0 values, i.e. m_{true} and $\sigma_{0,\text{true}}$, random strength data can be obtained from Eq. (17) provided random numbers between 0 and 1 are substituted for the probability of failure P. For the sake of convenience, we let m_{true}=10 and $\sigma_{0,\text{true}}$=1 throughout this study. A computer program was written, which used a sample of random numbers to obtain strength values σ_1,σ_2,...,σ_i,...,σ_n. This set of strength values was regarded as a fictitious sample, and then analyzed with each of the 14 methods listed in Table 1 to give the estimated values of the Weibull parameters.

For illustration of the effect of the sample size, the generated random samples were of size n increased progressively from 5 to 100. The above simulation procedure was repeated 10,000 times; therefore, a total of 10,000 samples were generated and 10,000 sets of the estimated Weibull modulus and scale parameter were obtained for each sample size and each method. Then the mean value $\bar{m}$, the standard deviation S_m, and the mean square error MSE_{m} of these Weibull moduli were computed from

$$\bar{m} = \sum_{j=1}^{10^4}\frac{m_j}{10^4} \tag{18}$$

Table 1 Estimation methods under investigation

Method	Type	Equation for W_i	Equation for P_i
1	LR	–	3a
2	LR	–	3b
3	LR	–	3c
4	LR	–	3d
5	WLR_1	9a	3a
6	WLR_1	9a	3b
7	WLR_1	9a	3c
8	WLR_1	9a	3d
9	WLR_2	9b	3a
10	WLR_2	9b	3b
11	WLR_2	9b	3c
12	WLR_2	9b	3d
13	ML	–	–
14	Moments	–	–

$$S_{\mathrm{m}}^2 = \sum_{j=1}^{10^4} \frac{\left(m_j - \bar{m}\right)^2}{10^4 - 1} \tag{19}$$

$$MSE_{\mathrm{m}} = \sum_{j=1}^{10^4} \frac{\left(m_j - m_{\mathrm{true}}\right)^2}{10^4} \tag{20}$$

where m_j is the estimated Weibull modulus of the jth sample. The coefficient of variation CV_m of the Weibull modulus was obtained from S_m divided by $\bar{m}$.

It should be pointed out that, for the LR, WLR and ML methods, although the above simulation was carried out for the arbitrarily chosen values of m_{true}=10 and $\sigma_{0,\mathrm{true}}$=1, its results are valid for any value of m_{true} and $\sigma_{0,\mathrm{true}}$, since previous studies have shown that the estimated m/m_{true} is distributed independently of the prescribed values of m_{true} and $\sigma_{0,\mathrm{true}}$ [7, 8, 10, 22, 23]. However, this property does not hold for the moments method.

Results and discussion

Mean value and coefficient of variation

Figure 1 shows the dependence of the normalized mean values of the estimated Weibull moduli, $\bar{m}/m_{\mathrm{true}}$, on the sample size n for the 14 methods investigated. For the sake of easy comparison and good clarity, the results obtained with various types of the estimation methods are plotted in one chart where the axes of each subfigure have the same scale. For an unbiased estimate, $\bar{m}/m_{\mathrm{true}}$ is expected to be close to unity.

It can be seen that for all methods, the estimators of the Weibull modulus are always biased, especially for smaller sample sizes. In most cases the bias increases rapidly as the sample size decreases. Previous authors [7, 8] have noticed a similar phenomenon for their selected methods. Next, the probability estimators have great effects on the bias of the estimated Weibull modulus. In the LR, WLR_1 and WLR_2, Eq. (3a) gives the least-biased estimate of the Weibull modulus for $n\geq 20$. The next is Eq. (3d), followed by Eq. (3c). The estimator given by Eq. (3b) leads to the largest bias for all sample sizes examined. Finally, the weight factors have also effects on the bias of the estimated values. The two weight factors improve the results of the estimators, Eqs. (3b–d); however, there is no significant improvement on the results of the estimator, Eq. (3a) for $n\geq$ 20. These results are in agreement with those reported by previous authors [8, 9, 11, 12, 15, 17].

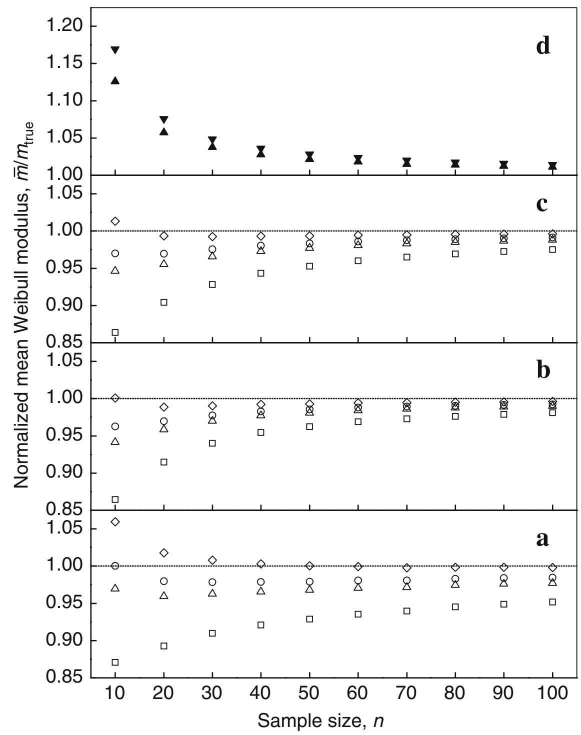

Fig. 1 Estimated Weibull modulus as a function of the sample size for the LR (**a**), WLR_1 (**b**), WLR_2 (**c**), and ML and Moments (**d**). (◇) Eq. (3a), (□) Eq. (3b), (△) Eq. (3c), (○) Eq. (3d), (▼) ML, (▲) Moments

In Fig. 2, the coefficient of variation of the estimated Weibull modulus is plotted as a function of the inverse square root of the sample size. The equation for the dotted line in each subfigure is $CV_{\mathrm{m}}=1/n^{1/2}$, used as a reference. The results of the LR, ML and moments methods are close to those given by Khalili and Kromp [7], Bergman [8] and Trustrum and Jayatilaka [10]. Part of the results of the WLR_1 and WLR_2 are also in agreement with those reported by previous authors for their selected methods [11–13].

Clearly, the coefficient of variation is as expected decreasing with increasing sample size for all methods. In

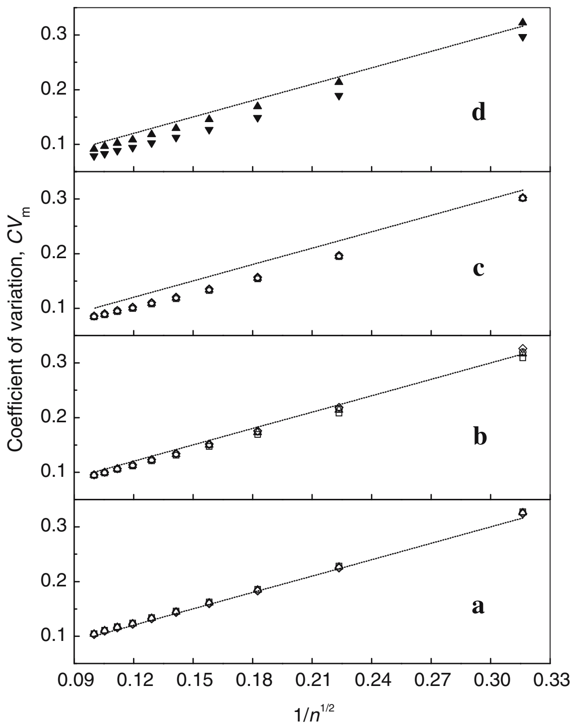

Fig. 2 Dependence of the coefficient of variation of the estimated Weibull modulus on the inverse square root of the sample size for the LR (**a**), WLR_1 (**b**), WLR_2 (**c**), and ML and Moments (**d**). (◇) Eq. (3a), (□) Eq. (3b), (△) Eq. (3c), (○) Eq. (3d), (▼) ML, (▲) Moments

each of the LR, WLR_1 and WLR_2, the coefficients of variation for different probability estimators are approximately equal at any sample size; thereby the probability estimators have no significant effects on the coefficient of variation. Similar results have also been reported by Khalili and Kromp [7], Bergman [8] and Trustrum and Jayatilaka [10]. However, the weight factors have great effects on the coefficient of variation. Figure 2 indicates that, at various sample sizes the data points for the LR are always slightly higher than its reference line; those for the WLR_1 are slightly lower than its reference line; however, those for the WLR_2 are considerably lower than its reference lines. On a whole, at a given sample size the coefficients of variation for various types of the methods follow the sequences:

$$CV_{\mathrm{LR}} > CV_{\mathrm{WLR\_1}} > CV_{\mathrm{WLR\_2}} > CV_{\mathrm{ML}} \quad \text{and } CV_{\mathrm{Moments}} > CV_{\mathrm{ML}}.$$

In the literature [7, 10, 13], the estimation precision of the Weibull modulus was usually judged by the coefficient of variation. The smaller the coefficient of variation is, the higher the estimation precision is. Based on such a criterion, previous authors concluded that the ML method leads to the highest estimation precision of the Weibull modulus, which has been recommended by several publications [1, 7, 13]. As for the methods based on the linear regression analysis, the use of the weight factors leads to a more accurate estimation of the Weibull modulus. Especially the weight factor given by Eq. (9b) is better and method 11 comes first among the LR, WLR_1 and WLR_2 [13]. In addition, from the biasing results shown in Fig. 1, previous studies also came to the conclusion that in the LR method, the probability estimator given by Eq. (3a) is the best, while Eq. (3b) gives the worst estimation of the Weibull modulus [7–9, 15, 17].

Criterion for the comparison of the estimation methods

Statistics textbooks tell us that the higher the probability of computing an estimate near to the true value is, the higher the estimation precision is. For an unbiased estimate, the variance (or the standard deviation) may be used as a criterion for the estimation precision. The smaller the variance is, the higher the estimation precision is. However, for a biased estimate, the variance criterion becomes meaningless, since it only reveals the probability of computing an estimate near to the expected value, i.e. mean value. If the expected value is far away from the true value, a poor estimation is certain to occur even if the variance is very small. In this case, the mean square error was recommended by statisticians as the criterion for the estimation precision [24].

By mathematical transformation, Eq. (20) can be rewritten as

$$MSE_{\mathrm{m}} = \frac{10^4 - 1}{10^4} \cdot S_{\mathrm{m}}^2 + (\bar{m} - m_{\mathrm{true}})^2 \tag{21}$$

Clearly, the mean square error is composed of two parts: the variance and the square of the bias. The smaller the sum of the two parts is, the higher the estimation precision is. It is also seen that for an unbiased estimate, the mean-square-error criterion is equivalent to the variance criterion.

As shown in Fig. 1, for all methods investigated, the estimators of the Weibull modulus are always biased, especially for smaller sample sizes. It indicates that the estimation precision of the Weibull modulus is related not only to the bias but also to the variance. Therefore, the mean-square-error criterion should be used for the comparison of the estimation methods.

It should be mentioned that in statistics textbooks, the coefficient of variation has never been used as a criterion for the estimation precision. The coefficient of variation is the standard deviation divided by the mean value, which represents the relative breadth of the estimated data distribution with reference to the mean. The use of the coefficient of variation as the criterion will be highly advantageous to the estimation methods with a larger estimated $\bar{m}$, for example to the ML method, since a larger $\bar{m}$ results in a smaller coefficient of variation.

Table 2 Normalized square root of the mean square error, $\sqrt{MSE_m}/m_{\text{true}}$, of various estimation methods

n	Method													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
5	0.7023	0.5184	0.5949	0.6280	0.6356	0.5030	0.5580	0.5820	0.6481	0.5050	0.5651	0.5910	0.9155	0.8255
10	0.3487	0.3133	0.3175	0.3255	0.3267	0.2998	0.3038	0.3100	0.3068	0.2938	0.2900	0.2940	0.3863	0.3844
20	0.2293	0.2303	0.2210	0.2221	0.2165	0.2086	0.2091	0.2112	0.1957	0.2003	0.1917	0.1921	0.2174	0.2326
30	0.1843	0.1918	0.1813	0.1811	0.1742	0.1698	0.1703	0.1714	0.1559	0.1599	0.1537	0.1539	0.1633	0.1795
40	0.1600	0.1692	0.1591	0.1584	0.1505	0.1475	0.1480	0.1487	0.1347	0.1372	0.1330	0.1332	0.1366	0.1519
50	0.1432	0.1530	0.1434	0.1425	0.1335	0.1316	0.1318	0.1323	0.1198	0.1217	0.1184	0.1186	0.1190	0.1336
60	0.1316	0.1411	0.1323	0.1313	0.1226	0.1211	0.1213	0.1217	0.1099	0.1111	0.1088	0.1089	0.1075	0.1212
70	0.1215	0.1311	0.1226	0.1216	0.1130	0.1118	0.1119	0.1123	0.1015	0.1023	0.1005	0.1007	0.0982	0.1108
80	0.1152	0.1238	0.1162	0.1153	0.1062	0.1053	0.1054	0.1056	0.0957	0.0961	0.0948	0.0949	0.0918	0.1046
90	0.1088	0.1170	0.1099	0.1090	0.0996	0.0988	0.0989	0.0991	0.0898	0.0901	0.0891	0.0892	0.0857	0.0983
100	0.1027	0.1106	0.1039	0.1030	0.0949	0.0942	0.0943	0.0945	0.0857	0.0858	0.0849	0.0851	0.0813	0.0926

One of the possible reasons for the use of the coefficient of variation as the criterion by previous authors is that they considered the bias of the estimated Weibull modulus can be removed with a correction factor [8, 10, 13, 25, 26]. However, from analytical analyses and numerical calculations, Peterlik [27] pointed out that each set of strength data gives the statistically correct Weibull parameters and that the bias arises only from the method of adding the parameters, if one tries to obtain a mean value from a number of sets of strength data. In practice, when only one set of strength data is available, the correction factor should not be applied.

Comparisons based on the mean square error

Table 2 shows the normalized square root of the mean square error of the estimated Weibull modulus for the 14 methods and selected sample sizes. Clearly, for any method, as the sample size n increases the value of $\sqrt{MSE_{\text{m}}}/m_{\text{true}}$ decreases and hence the estimation precision of the Weibull modulus increases. It can also be seen that the probability estimators and the weight factors both have effects on the mean square error.

Table 3 gives the comparison between different probability estimators. It can be seen that except for the WLR_1 the ranking number of the probability estimators is highly sensitive to the sample size. In the LR, the estimator given by Eq. (3b) gives the most accurate estimation for $5\leq n\leq 10$; Eq. (3c) is the best for $11\leq n\leq 28$; Eq. (3d) is to be preferred for $29\leq n\leq 69$; however, Eq. (3a), recommended by several authors [7–9, 15, 17], comes first only for $n\geq 70$. In the WLR_1, the estimator, Eq. (3b), always leads to the highest precision of estimation for all sample sizes examined. In the WLR_2, Eq. (3c) is the most successful estimator for $n\geq 9$, and Eq. (3b) for $5\leq n\leq 8$.

In Table 4 the comparison between different weight factors is shown. It is clear that for any probability estimator, the weight factor given by Eq. (9b) results in the most accurate estimation of the Weibull modulus for $n\geq 7$, followed by the weight factor, Eq. (9a). However, the LR without the weight factor always gives the worst estimation for any probability estimator and any sample size.

Table 5 gives the comparison between the ML method and the others. It can be seen that for smaller sample sizes the ML method leads to a poorer estimation of the Weibull modulus than the others. For example, for $n\leq 52$ the WLR_2 with an appropriate probability estimator is better than the ML. It arises from the fact that the bias of the Weibull modulus estimated with the ML method increases rapidly as the sample size decreases, as shown in Fig. 1. These results indicate that for smaller sample sizes the smallest coefficient of variation does not satisfactorily compensate for a large bias for the ML method.

From Tables 2–5, it is shown that among the 14 methods investigated, method 6 gives the most accurate estimation of the Weibull modulus for n=5; method 10 is the best for $6\leq n\leq 8$; method 11 is to be preferred for $9\leq n\leq 52$; method 13, i.e. the ML method recommended to be used for all sample sizes by previous studies [1, 7, 13], comes first only for $n\geq 53$.

Table 3 Order number of the probability estimators ranked based on the estimation precision of the Weibull modulus

Type	Sample size	Probability estimator			
		Eq. (3a)	Eq. (3b)	Eq. (3c)	Eq. (3d)
LR	$5\leq n\leq 10$	4	1	2	3
	$11\leq n\leq 12$	4	2	1	3
	$13\leq n\leq 18$	4	3	1	2
	$19\leq n\leq 28$	3	4	1	2
	$29\leq n\leq 49$	3	4	2	1
	$50\leq n\leq 69$	2	4	3	1
	$70\leq n\leq 100$	1	4	3	2
WLR_1	$5\leq n\leq 100$	4	1	2	3
WLR_2	$5\leq n\leq 8$	4	1	2	3
	$9\leq n\leq 10$	4	2	1	3
	$11\leq n\leq 12$	4	3	1	2
	$13\leq n\leq 100$	3	4	1	2

Table 4 Order number of the weight factors ranked based on the estimation precision of the Weibull modulus

Probability estimator	Sample size	Weight factor		
		–	Eq. (9a)	Eq. (9b)
Eq. (3a)	$5\leq n\leq 6$	3	1	2
	$7\leq n\leq 100$	3	2	1
Eq. (3b)	n=5	3	1	2
	$6\leq n\leq 100$	3	2	1
Eq. (3c)	$5\leq n\leq 6$	3	1	2
	$7\leq n\leq 100$	3	2	1
Eq. (3d)	$5\leq n\leq 6$	3	1	2
	$7\leq n\leq 100$	3	2	1

Table 5 Range of the sample size where the ML method is superior to the others

Type	Sample size
LR	$n\geq 19$
WLR_1	$n\geq 24$
WLR_2	$n\geq 53$
Moments	$n\geq 11$

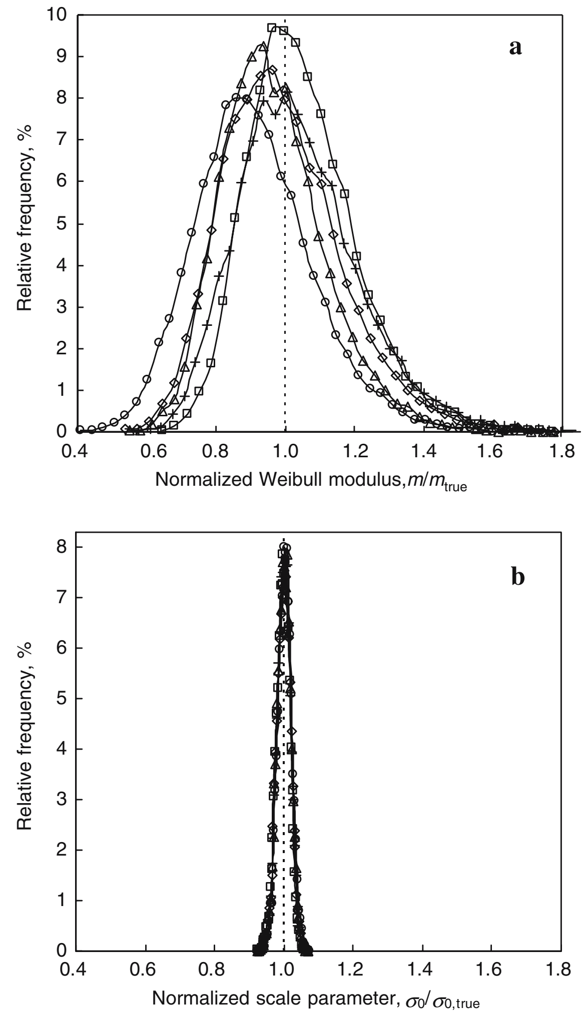

Fig. 3 Probability density distributions of the estimated Weibull modulus (**a**) and scale parameter (**b**) at a sample size n=30. (○) Method 2, (◇) Method 5, (△) Method 11, (□) Method 13, (+) Method 14

Estimation of the scale parameter

Using Monte Carlo simulations, the 10,000 estimated values of the scale parameter σ_0 were produced for each sample size and each method. The mean value, the standard deviation and the mean square error were calculated. And then the 10,000 estimated σ_0 were ranked in ascending order and divided into 40 equally sized intervals. The number of estimated σ_0 falling into each interval was counted. This number, normalized through division by 10,000, the total number of estimated σ_0, produces the relative frequency of occurrence, which was taken as the y-value. The midpoint of the given interval was used as the x-value. The resulting histogram can be regarded as an empirical probability density distribution [7]. For the sake of comparison, the same data processing was also carried out for the estimated Weibull moduli.

As an example, the probability density distributions of the estimated Weibull modulus and scale parameter at a sample size of 30 for 5 selected methods are shown in Fig. 3, where the x-axes of two subfigures have the same range and the same scale. The results of other sample sizes and other methods are similar to those shown in Fig. 3.

It can be seen that all methods give a similar distribution of $\sigma_0/\sigma_{0,\mathrm{true}}$ that scatters in the vicinity of the true value in a much smaller range, as compared with the Weibull modulus. Simulations reveal that for any sample size and any method, the normalized square root of the mean square error of the estimated scale parameter is about one tenth of that of the estimated Weibull modulus; therefore, the scale parameter can be estimated with accuracy about an order of magnitude higher than the Weibull modulus [7, 18].

From Fig. 3, it may be notable that the distribution of m/m_{true} is asymmetrical and lightly skewed to the right; the distribution of $\sigma_0/\sigma_{0,\mathrm{true}}$, however, takes an approximately symmetrical form. Similar results have also been reported by Khalili and Kromp [7]. For any sample size and any method, the mean value of the estimated scale parameter is always close to its true value, and its bias is negligible. Therefore, the estimator of the scale parameter obtained with any method is approximately unbiased.

Safety factor

From the point of view of material sciences, an accurate estimation of the Weibull parameters is pursued [7–18]. However, from an engineering point of view, the safety is of the first importance, while the estimation precision is the

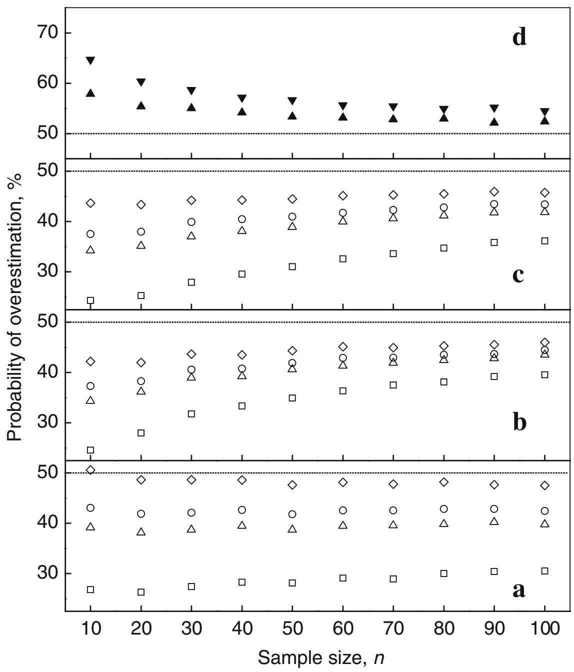

Fig. 4 Occurrence probability of the Weibull modulus overestimated as a function of the sample size for the LR (**a**), WLR_1 (**b**), WLR_2 (**c**), and ML and Moments (**d**). (◇) Eq. (3a), (□) Eq. (3b), (△) Eq. (3c), (○) Eq. (3d), (▼) ML, (▲) Moments

second. For the prediction of the probability of failure at low stresses or the fracture stress at low probabilities of failure, the Weibull modulus has a more effect than the scale parameter does. This can be visualized by plotting the Weibull curve according to Eq. (1). Whilst the scale parameter moves the curve left and right, the Weibull modulus rotates the curve. An overestimation of the Weibull modulus often leads to an underestimation of the probability of failure at low stresses or an overestimation of the fracture stress at low probabilities of failure, and hence a lower safety will arise in reliability prediction.

In Fig. 4, the occurrence probability of the Weibull modulus overestimation, i.e. $m/m_{\mathrm{true}}>1$ is plotted as a function of the sample size. The higher the probability is, the lower the safety is. Clearly, the probability for the ML method is the highest, which is always larger than about 55% in the range examined, followed by the moments method. It implies that for the two methods, the overestimation of the Weibull modulus occurs more frequently than its underestimation [7, 28]. In addition, the probability of overestimation for the two method increases as the sample size decreases. However, the probabilities of overestimation for the LR, WLR_1 and WLR_2 methods are always smaller than 50%. Thus it can be seen that these methods result in a higher safety than the ML and moments methods.

Conclusions

Monte Carlo simulations reveal that the estimated scale parameter is approximately unbiased and has a much higher accuracy than the Weibull modulus. However, the estimator of the Weibull modulus is always biased, especially for smaller sample sizes. The mean square error is therefore used as a criterion for the comparison of the estimation precision of different methods.

It is shown that in the LR and WLR_2, the preferred probability estimator varies with the sample size; however, in the WLR_1, the most successful estimator is given by Eq. (3b) for all sample sizes. Also, in the regression analysis, the use of the weight factors improves the estimation precision of the Weibull modulus. The weight factor given by Eq. (9b) is to be preferred for $n\geq 7$. It is concluded that method 6 gives the most accurate estimation of the Weibull modulus for n=5; method 10 is the best for $6\leq n\leq 8$; method 11 is to be preferred for $9\leq n\leq 52$; method 13, i.e. the ML method, comes first only for $n\geq 53$. Moreover, it is reaffirmed that for any method, the accuracy of the estimated Weibull modulus increases with increasing sample size.

The estimation methods were also compared from an engineering point of view. It is found that the ML and moments methods lead to the overestimation of the Weibull modulus more frequently than its underestimation, and hence with a lower safety than the LR, WLR_1 and WLR_2 methods.

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